LABORATORY STUDIES IN ASSET TRADING:

PART III--Stochastic Market Structures*

by

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INTRODUCTION

In this report we investigate the impact of alternative market structures in an environment of stochastic demand and supply. Our method shall be to begin with the very simplest structures and build towards the richness found in real-world markets.

To characterize the exact nature of stochastic demand and supply, we adopt the assumptions of the previous report in this series. We further assume the existence of "orders" which represent occurrences at discrete points in continuous time, generated by market participants as concrete embodiments of their individual willingness to buy or to sell the asset in question. To be consistent with contract law, we take orders to mean firm offers to buy or sell or sell under specified conditions, good until cancelled or withdrawn, which may result in completed contracts (transactions) when a buyer and seller agree on conditions. Market structure is taken to mean (1) the conditions which are permitted in orders, and (2) the rules and procedures of effecting transactions in a multiple-order situation. We classify orders into two groups, buy orders and sell orders.

More formally, an order ϕ_i will be characterized as a triple $\phi_i = (p_i, q_i, c_i)$ where p_i is a "price," or amount of a numeraire asset; q_i is a "quantity," or amount of the asset in question; and c_i is a vector of conditions which apply to that order. To characterize the state of a market at time t define two sets $B(t)$ and $S(t)$, representing the buy orders and sell orders respectively, which are active at t. Let $R(t)$ be a vector of variables which otherwise characterize the market state at time t (e.g., the sign of the "tick" on the NYSE). Then the market state at time t is summarized by the triple $M(t) = \big(B(t), S(t), R(t)\big)$.

According to the assumptions of the previous report, order arrivals for inclusion in the sets B(t) and S(t) will follow a Poisson process. To simplify things, we shall later assume that cancellations, withdrawals, etc. (which we uniformly term order "extinctions"), obey a memoryless process so that the markets we will consider herein will possess Markovian properties.

MODEL I: BENEVOLENT MARKET-MAKER, INFINITE INVENTORIES

In this first and simplest market structure model, we shall assume that:

(I.1) Arrivals of buy orders and sell orders are Poisson distributed in time, with stationary rate functions $\lambda_B(p)$ and $\lambda_S(p)$ that depend only on price p. q is assumed equal to 1, and the condition vectors c are assumed null.

(I.2) The stochastic equilibrium price is known to all market participants, all bids and offers are made at that price. I.e., the selection functions for supply and demand are point-mass functions at the stochastic equilibrium price, a term defined below.

(I.3) All exchanges are made through a single central "market-maker," who possesses a monopoly on all trading. No direct exchanges between buyers and sellers take place.

(I.4) The central market-maker is a benevolent monopolist in the sense that he immediately accepts and executes all bids and offers at the equilibrium price.

(I.5) The central market-maker has infinite inventories of both assets, which we shall call "stock" and "cash."

(I.6) There are no transactions costs.

By way of explanation, the stochastic equilibrium price means the following: The mean value functions of the stochastic demand and supply (see formula (5) of Part II) are here independent of t, and are in fact the rate parameters λ_B and λ_S (functions of price only) of the Poisson order arrival processes. To diagram the situation, we have:

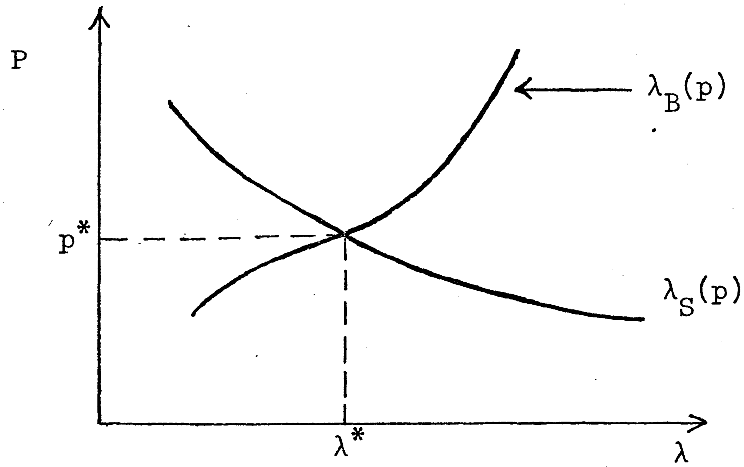

Figure 1

where p^* is the stochastic equilibrium price, λ^* is the stochastic equilibrium rate, assuming as usual that these "demand and supply" functions intersect.

Clearly, there are no nontrivial price or order-queuing implications of Model I. Volume, i.e., the amount of stock exchanged in a given interval of time, is only slightly interesting. Since at the equilibrium price p^*, we have $\lambda_B(p^*) = \lambda_S(p^*) = \lambda^*$, volume is Poisson distributed with rate $2\lambda^*$. In a time interval $[t, t+\tau]$, the expected volume is thus $2\lambda^*\tau$, the variance is $4\lambda^{*2}\tau^2$.

MODEL II: THE BENEVOLENT MARKET-MAKER, FINITE INVENTORIES

In this model we adopt the following assumptions, where the notation "$\Longleftrightarrow$" means "is identical to":

(II.1) $\Longleftrightarrow$ (I.1) (Stationary rates, $q = 1$, c null)

(II.2) $\Longleftrightarrow$ (I.2) (Equilibrium price sole price)

(II.3) $\Longleftrightarrow$ (I.3) (Trading monopoly)

(II.4) $\Longleftrightarrow$ (I.4) (Benevolence)

(II.5) The market-maker has finite inventories I_S and I_c of stock and cash.

(II.6) $\Longleftrightarrow$ (I.6) (No transactions costs)

Note that only Assumption II.5 is different from that of Model I. Thus our interest lies in the affect of finite inventories. We shall term the market-maker "ruined" if either of his inventories are exhausted, since he is then unable to continue in his role.

First, it is clear that ultimate ruin is a certainty for the market-maker. What is of interest is the time until ruin and its nature. By analyzing the embedded random walk of his inventories via the usual methods, we find that the market-maker will be ruined by a stock depletion with probability $I_c(p^* I_S + I_c)^{-1}$ or by a cash depletion with probability $p^* I_s(p^* I_s + I_c)^{-1}$. The expected time until ruin is $I_s I_c / 2\lambda^* p^*$. (It is interesting to apply this latter formula to actual exchange liquidity requirements, for example, the NYSE requirement that its specialists be able to take a position of 400 shares of the securities in which they specialize. Assuming that 95% leverage is available, daily volume in his security is 6000 shares (the average), and that the specialist participates in about 25% of these trades, we see that the maximum of this quantity is about 53 days. Even if the approximations of the model to the real world are poor, the order of magnitude here makes it clear that specialists must pursue a policy of relating their prices to their inventories to avoid ruin; it is not the case that they simply respond to temporary fluctuations in demand and supply, as the stock exchange propaganda would suggest. Also, we see the importance of leverage: without it, the expected time to ruin is about 1/2 day.)

MODEL III: THE MONOPOLISTIC MARKET-MAKER

The next situation assumes the following:

(III.1) $\iff$ (II.1) (Stationary rates, $q = 1$, c null)

(III.2) All buy and sell orders are made at the prices set by the market-maker, p_B and p_S. That is, the selection function for demand is point-mass at p_B, that for supply is point-mass at p_S.

(III.3) $\iff$ (II.3) (Trading monopoly)

(III.4) The market-maker maximizes expected profit per unit time by buying at one price, selling at another.

(III.5) The market-makers' inventories are essentially infinite but subject to a no-drift policy condition, explained below.

(III.6) $\iff$ (II.6) (No transactions costs.)

Here the central market-maker takes advantage of his monopolistic trading situation (III.3) to "buy low, sell high." The no-drift assumption means that the stochastic processes representing inventory levels must be martingales. Hence if p_B and p_S are the prices at which the market-maker will accept

$\bar{p}_S = \frac{\bar{\lambda} - \alpha}{\beta}$ and $\bar{p}_B = \frac{\gamma - \bar{\lambda}}{\delta}$, the spread (see Part IV) is $\bar{S} = \bar{p}_B - \bar{p}_S$, and the expected profit $\bar{\pi} = \bar{s}\bar{\lambda}$ per unit time. Relaxing Assumption (III.6), similar calculations including a transactions cost c per trade paid by the market-maker yield corresponding results where $\bar{\lambda}$ above becomes $\bar{\bar{\lambda}} = \bar{\lambda} - \frac{\beta\delta c}{(\beta + \delta)}$. If expected profit maximization, no-drift inventory policy, and approximate linearity of the order rates are realistic, we see the following behavioral hypotheses are suggested by Model III's treatment of the monopolistic market-maker: (1) He will establish two prices, one for buying $\bar{p}_S$, and one for selling, $\bar{p}_B$. (2) The spread can be either too low or too high in serving his interests, i.e., he will not use his monopolistic position to increase the spread beyond all limits. (3) Imposition of a transaction cost c on the market-maker will decrease his buying price and increase his selling price linearly with respect to c ; volume will also fall linearly with respect to c .

MODEL IV: MONOPOLISTIC MARKET-MAKER, FINITE INVENTORIES

It is of some interest to explore the impact of initial inventory limitations on the monopolistic market-maker, since now he may use his profits to increase his inventories. To this end, we let $I_s(t)$ and $I_c(t)$ be his inventories of stock and cash at time t , and replace the assumptions of the previous model by:

(IV.1) $\Longleftrightarrow$ (III.1) (Stationary rates, q = 1, c null)

(IV.2) $\Longleftrightarrow$ (III.2) (Point-mass selection functions)

(IV.3) $\Longleftrightarrow$ (III.3) (Monopoly)

(IV.4) $\Longleftrightarrow$ (III.4) (Profit maximizer)

(IV.5) At time 0 , the central market-maker has cash and stock inventories of $I_c(0)$ and $I_s(0)$, respectively. Subsequent negative inventories imply the market-makers' "ruin," i.e., inability to continue in his role.

(IV.6) <=> (III.6) (No transaction costs)

Let $N_B(t)$ and $N_0(t)$ be the numbers of bids and offers, respectively, executed by time t. Inventories are governed by the relationships

$$I_c(t) = I_c(0) + p_B N_B(t) - p_S N_S(t)$$

and (1)

$$I_s(t) = I_s(0) + N_S(t) - N_B(t) \quad .$$

Let $Q_k(t)$ be the probability that $I_c(t) = k$ and let $R_k(t)$ be the probability that $I_S(t) = k$. If we assume that $I_c(0)$ is much larger than either p_S and p_B, then the dynamics of cash inventory can be approximately described by a birth and death process via the differential equations

$$Q_k'(t) = -\left\{ p_B\lambda_B(p_B) + p_S\lambda_S(p_S)\right\} Q_k(t) + p_B\lambda_B(p_B)Q_{k-1}(t)$$
$$+ p_S\lambda_S(p_S)Q_{k+1}(t) \quad , \quad k \geq 1$$

and (2)

$$Q_0'(t) = p_S\lambda_S(p_S)Q_1(t)$$

where the initial conditions are

$$Q_k(0) = \begin{cases} 1 & \text{if } k = I_c(0) \\ 0 & \text{otherwise .} \end{cases} \qquad (3)$$

Analyzing the embedded Markov chain in this stochastic process by the usual methods yields the approximate ruin probabilities

$$\lim_{t \to \infty} Q_o(t) = \begin{cases} \left(\dfrac{p_S \lambda_S(p_S)}{p_B \lambda_B(p_B)}\right)^{I_c(0)} & \text{if } p_B\lambda_B(p_B) > p_S\lambda_S(p_S) \\ 1 & \text{otherwise.} \end{cases} \qquad (4)$$

Similarly (and this time without approximation)

$$\lim_{t \to \infty} R_o(t) = \begin{cases} \left(\dfrac{\lambda_B(p_B)}{\lambda_S(p_S)}\right)^{I_s(0)} & \text{if } \lambda_B(p_B) < \lambda_S(p_S) \\ 1 & \text{otherwise} \end{cases} \qquad (5)$$

From (4) and (5), we see that to avoid the certainty of eventual ruin, the central market-maker must set p_B and p_S so as to satisfy the simultaneous conditions

$$p_B\lambda_B(p_B) > p_S\lambda_S(p_S)$$

and $\qquad$ (6)

$$\lambda_S(p_S) > \lambda_B(p_B) \; ,$$

provided this is possible. In other words, the central market-maker with finite inventories must tread the narrow policy path given by (6); even in doing so, he risks some probability, given by (4) and (5), of ruin regardless.

The riskiness of this environment again compels us to suspect that the rational market-maker will adjust p_B and p_S <u>depending on the state of his inventories</u> in order to limit his ruin probabilities. Other hypotheses which might be drawm from Model III are: (a) <u>Ceteris paribus</u>, Equation (4) suggests that increasing the spread is one way of limiting the risk of cash inventory ruin.

(b) Assuming $\lambda_B(p_B) \approx \lambda_S(p_S)$, Equation (5) suggests that changing the ratio λ_B/λ_S slightly will be more effective than increased initial inventory in protecting against stock-out ruin.

MODEL V: THE CLEARING-HOUSE MARKET

In this model, we dispense with the central market-maker altogether. Since, however, order executions cannot then necessarily occur instantly upon arrival, we shall provide that all orders have a stochastic "lifetime." Specifically, we assume:

(V.1) $\Longleftrightarrow$ (IV.1) (Stationarity, $q = 1$, c null)

(V.2) All buy and sell orders are made at some known price p . (Point-mass selection functions both at p .)

(V.3) All transactions are made by "crossing" active orders, i.e., matching a buy order with a sell order whenever possible.

(V.4) All orders have memoryless (exponential) "lifetimes" during which they are active, i.e., available for crossing.

(V.5) Null

(V.6) $\Longleftrightarrow$ (I.6) (No transactions costs)

We shall describe the current "state" of this market by an integer from the set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ in the following fashion: state "n" means that n buy orders are active in the market if $n > 0$ and $-n$ sell orders are active if $n < 0$. (Naturally, active bids and offers cannot simultaneously co-exist since there is but one price, and crossing takes place without delay. Thus the state description scheme given is complete.) We shall let q_{ij} denote the state transition intensities from state i to state j (see, e.g., Parzen, Stochastic Processes). Assuming that the lifetimes of orders are identically expontentially distributed with rate ν (the "death" rate), we have the following transition intensities between states:

$$q_{n,n+1} = \begin{cases} \lambda_B(p) & \text{if} \quad n \geq 0 \\ \lambda_B(p) - n\nu \,, & n < 0 \end{cases}$$

$$q_{n,n-1} = \begin{cases} \lambda_S(p) + n\nu \,, & n > 0 \\ \lambda_S(p) \,, & n \leq 0 \end{cases} \tag{6}$$

$$q_{n,n} = \begin{cases} \lambda_B(p) + \lambda_S(p) + n\nu \,, & n > 0 \\ \lambda_B(p) + \lambda_S(p) \,, & n = 0 \\ \lambda_B(p) + \lambda_S(p) - n\nu \,, & n < 0 \end{cases}$$

After the transient behavior (which depends upon the initial state of the market) is gone, the probability of finding the market in a particular state is given by the equilibrium probabilities a_n as follows:

$$a_n = \begin{cases} a_o \prod_{i=1}^{n} \dfrac{\lambda_B(p)}{\lambda_S(p) + i\nu} & \text{for} \quad n \geq 1 \\ a_o \prod_{i=1}^{-n} \dfrac{\lambda_S(p)}{\lambda_B(p) + i\nu} & \text{for} \quad n \leq -1 \end{cases} \tag{7}$$

where a_o is the equilibrium probability of finding no orders in the market. It is of interest to find a_o , $a_+ = \sum_{i=1}^{\infty} a_i$, and $a_- = \sum_{i=-\infty}^{-1} a_i$ which are related by the norming equation

$$a_o + a_+ + a_- = 1 \,. \tag{8}$$

From (7) we have

$$a_+ = \sum_{n=1}^{\infty} a_o \prod_{i=1}^{n} \frac{\lambda_B(p)}{\lambda_s(p)+i\nu}$$

$$= a_o \sum_{n=1}^{\infty} \left(\frac{\lambda_B(p)}{\nu}\right)^n \prod_{i=1}^{n} \frac{1}{\left(\frac{\lambda_s(p)}{\nu}\right)+i} \quad . \tag{8}$$

To simplify the calculations, let $\chi = \left(\frac{\lambda_B(p)}{\nu}\right)$ and $y = \left(\frac{\lambda_s(p)}{\nu}\right)$.

Then

$$a_+ = a_o \sum_{n=1}^{\infty} \chi^n \prod_{i=1}^{n} \frac{1}{(y+i)} \quad . \tag{9}$$

Multiplying both sides of (9) by χ^y yields

$$\rho \equiv \chi^y a_+ = a_o \sum_{n=1}^{\infty} \chi^{n+y} \prod_{i=1}^{n} \frac{1}{(y+i)} \tag{10}$$

from whence we see that ρ satisfies the differential equation

$$\frac{d\rho}{d\chi} = \rho + a_o \chi^y \quad . \tag{11}$$

The solution of (11) is obtained via the use of the integrating factor $e^{-\chi}$ (see, e.g., Rainville, <u>Elementary Differential Equations</u>, p. 41), and thus

$$\rho = \frac{a_o\Gamma(y,\chi) - C}{e^{-\chi}} \tag{12}$$

where $\Gamma(y,\chi) = \int_0^{\chi} t^y e^{-t} dt$ and C is a constant of integration.

Therefore we have

$$a_+ = \frac{\rho}{\chi^y} = \frac{a_o \Gamma(y,\chi) - C}{\chi^y e^{-\chi}} . \tag{13}$$

Now $\lim_{\chi \to 0} a_+ = \lim_{\chi \to 0} \frac{a_o \Gamma(y,\chi) - C}{\chi^y e^{-\chi}} = 0$, since $\Gamma(y,\chi) \approx \chi e^{-\chi} \chi^{y-1}$ for small χ . This implies that $C = 0$. Therefore

$$a_+ = \frac{a_o \Gamma(y,\chi)}{\chi^y e^{-\chi}} = \frac{a_o \Gamma\left(\frac{\lambda_S(p)}{\nu}, \frac{\lambda_B(p)}{\nu}\right)}{\left(\frac{\lambda_B(p)}{\nu}\right)^{\left(\frac{\lambda_S(p)}{\nu}\right)} e^{-\left(\frac{\lambda_B(p)}{\nu}\right)}} \tag{14}$$

In analogous fashion,

$$a_- = \frac{a_o \Gamma(\chi,y)}{y^{\chi} e^{-y}} = \frac{a_o \Gamma\left(\frac{\lambda_B(p)}{\nu}, \frac{\lambda_S(p)}{\nu}\right)}{\left(\frac{\lambda_S(p)}{\nu}\right)^{\left(\frac{\lambda_B(p)}{\nu}\right)} e^{-\left(\frac{\lambda_B(p)}{\nu}\right)}} . \tag{15}$$

Equations (8), (14), and (15) may then be solved for

$$a_o = \frac{\chi^y y^{\chi} e^{-(\chi+y)}}{\chi^y y^{\chi} e^{-(\chi+y)} + y^{\chi} e^{-y} \Gamma(y,\chi) + \chi^y e^{-y} \Gamma(\chi,y)} . \tag{16}$$

Example: Suppose $p = p^*$, the equilibrium price. Then by definition

$\lambda_B(p) = \lambda_S(p) = \lambda$. Suppose further that $\frac{\lambda}{\nu} = 2$, that is, the individual order arrival rates are each twice as large as the order extinction rate ν . (E.g., suppose that mean time between orders is 4 minutes, with each having a mean lifetime of 8 minutes.) Then straightforward calculations show that

$$a_0 \cong .294 \qquad a_+ = a_- = .353$$

$$a_{-1} = a_1 \cong .196$$

$$a_{-2} = a_2 \cong .098$$

$$a_{-3} = a_3 \cong .039$$

$$\vdots$$

The expectation of the market state is

$$E[n] = \sum_{n=-\infty}^{\infty} n \, a_n \ , \qquad (17)$$

is found via applying the transition intensities (6) in the Kolmogorov forward equations at equilibrium, namely

$$0 = -q_{nn} \, a_n + q_{n-1,n} \, a_{n-1} + q_{n+1,n} \, a_{n+1} \ , \qquad (18)$$

multiplying by n and summing:

$$0 = \sum_{n=-\infty}^{\infty} n \left\{-q_{nn} a_n + q_{n-1,n} a_{n-1} + q_{n+1,n} a_{n+1}\right\}$$

$$= \sum_{n=-\infty}^{\infty} \left\{-nq_{nn} a_n + (n+1)q_{n,n+1} a_n + (n-1) q_{n,n-1} a_n\right\} \qquad (19)$$

$$= \sum_{n=-\infty}^{\infty} \left(\lambda_B(p) - \lambda_S(p) - n\nu\right\} a_n \quad .$$

Thus we have

$$E[n] = \frac{\lambda_B(p) - \lambda_S(p)}{\nu} \quad . \qquad (20)$$

The expected number of orders awaiting execution is $E[|n|]$, which by a similar calculation is found to be

$$E[|n|] = \frac{\left(\lambda_S(p) + \lambda_B(p)\right)a_o + \left(\lambda_B(p) - \lambda_S(p)\right)a_+ + \left(\lambda_S(p) - \lambda_B(p)\right)a_-}{\nu} \qquad (21)$$

Thus at stochastic equilibrium, $\lambda^* = \lambda_B(p^*) = \lambda_S(p^*)$, we have $E[|n|] = 2\lambda^*/\nu$.

Among the hypotheses that might be drawn from this model are the following: (a) The probability of finding no queued orders in a market is well approximated by the formula (16); (b) The average number of queued orders is inversely proportional to the order extinction rate, as per (21); and (c) the frequency of finding a certain number of queued orders in the market has the quasi-geometric distribution of (7). Appropriate real-world markets for testing the model would seem to be, say, dealers in commodities for which it is difficult or impossible to maintain dealer inventories.

MODEL VI: THE PURE DOUBLE-AUCTION MARKET

In this model of market structure, we allow orders to take on several prices (multiple-point selection functions) and introduce order priority for the first time. Specifically, we assume:

(VI.1) Arrivals of buy and sell orders are Poisson distributed in time, with stationary rate functions $\lambda_B(p_j)$, $\lambda_S(p_j)$, $j = 1,2,\ldots,K$ that depend only on the prices $p_1 < p_2 < \ldots < p_K$. Order quantities q are assumed equal to 1, and the condition vectors c assumed null.

(VI.2) All orders are made at prices $p_1, p_2, \ldots, p_K$, except that buy orders which are made at a price greater than the current "ask" price, $\inf \{p_i | (p_i, q_i, c_i) \in S(t)\}$, are lowered to the ask price, and sell orders which are made at a price less than the current "bid" price, $\sup \{p_i | (p_i, q_i, c_i) \in B(t)\}$, are raised to the bid price.

(VI.3) All transactions are made by crossing active orders that agree on price.

(VI.4) $\Longleftrightarrow$ (V.4) (Exponential order lifetimes)

(VI.5) Null

(VI.6) $\Longleftrightarrow$ (V.6) (No transactions costs)

Our first objective will be to describe the dynamics of this "double-auction" market structure. We introduce the following notation:

$$\lambda \equiv \sum_{j=1}^{k} \lambda_B(p_j) \qquad b_j \equiv \frac{\lambda_B(p_j)}{\lambda}$$
$$\mu \equiv \sum_{j=1}^{k} \lambda_S(p_j) \qquad s_j \equiv \frac{\lambda_S(p_j)}{\mu} \tag{22}$$

(ν will again be the order extinction rate.)

Making use of the fact that a superposition of Poisson arrivals is again Poisson, λ and μ represent the total arrival rates of buy and sell orders, respectively. b_j is the probability that an incoming buy order has price p_j, s_j the corresponding probability for sell orders.

We describe the current state of the market as follows: for each price, we tally +1 for each of the buy orders at that price or -1 for each of the sell orders at that price. Let n_j be the derived number associated with price p_j. For example, Figure 3 depicts one possible market state for K=9:

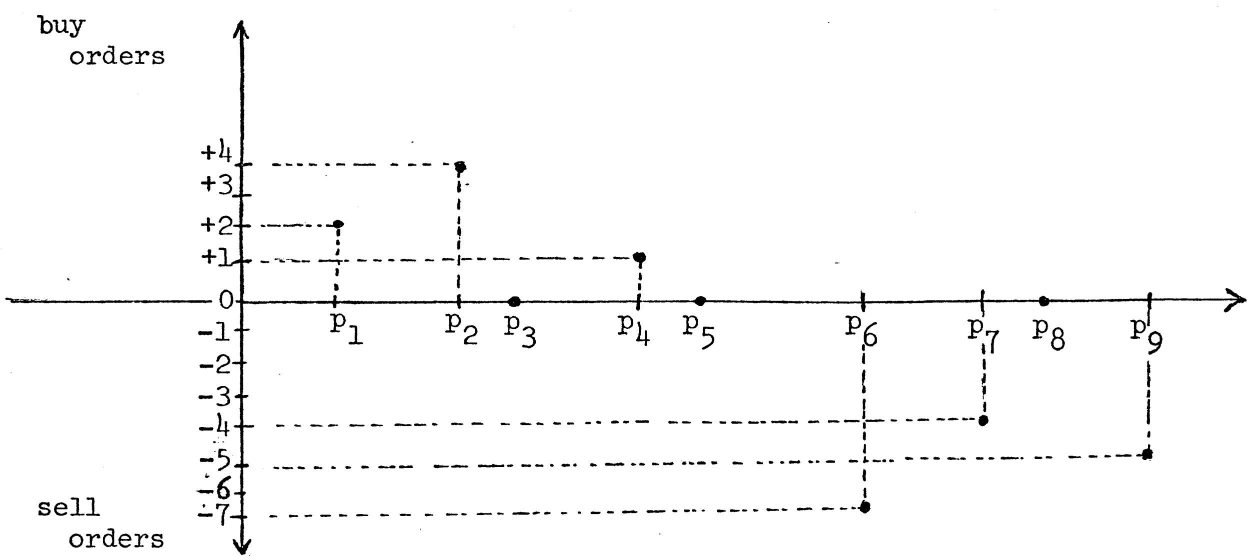

Figure 3

In Figure 3, p_4 is the bid price, p_6 is the ask price. If the next event were the arrival of a buy offer at price p_6 and $n_6 = -7$, a transaction would occur and n_6 would then have the value -6, etc. The market state is thus characterized by the vector $(n_1, n_2, \ldots, n_K)$. Let $P_{i_1,i_2,\ldots i_K}(t)$ be the probability that $(n_1, n_2,\ldots, n_K) = (i_1, i_2,\ldots,i_K)$ at time t and define

$$\gamma(i_1,i_2,\ldots,i_K) = \begin{cases} 0 & \text{if } i_j \le 0 \text{ , } j=1,2,\ldots,K \\ J & \text{if } i_J > 0 \text{ and } i_{J+k} \le 0 \text{ , } k=1,2,\ldots,K-J \end{cases}$$

and

$$\theta(i_1,i_2,\ldots,i_K) = \begin{cases} K+1 & \text{if } i_j \ge 0 \text{ , } j=1,2,\ldots,K \\ J & \text{if } i_J < 0 \text{ and } i_{J-k} \ge 0 \text{ , } k=1,2,\ldots,J-1 \end{cases} \quad . \tag{23}$$

Put simply, $\gamma(i_1,i_2,\ldots,i_K)$ is the index of the bid price, and $\theta(i_1,i_2,\ldots,i_K)$ is the index of the ask price, for the market state description $(n_1,n_2,\ldots,n_K) = (i_1,i_2,\ldots,i_K)$, which we abbreviate as just γ and θ. We may then write the Kolmogorov backward equations of the market model:

$$\begin{aligned} \frac{\partial P_{i_1,i_2,\ldots,i_K}(t)}{\partial t} = & -\lambda - \mu - \nu \sum_{j=1}^{K} |i_j| \; P_{i_1,i_2,\ldots,i_K}(t) \\ & + \lambda \sum_{j=1}^{\theta} \{ b_j \; P_{i_1,i_2,\ldots,(i_j+1),\ldots,i_K}(t) \} \\ & + \lambda \Big(\sum_{j=\theta}^{K} b_j \Big) \; P_{i_1,i_2,\ldots,(i_\theta-1),\ldots,i_K}(t) \\ & + \mu \sum_{j=\gamma+1}^{K} s_j \; P_{i_1,i_2,\ldots,(i_j-1),\ldots,i_K}(t) \\ & + \mu \Big(\sum_{j=1}^{\gamma} s_j \Big) \; P_{i_1,i_2,\ldots,(i_\gamma+1),\ldots,i_K}(t) \\ & + \nu \sum_{j=1}^{\gamma} |i_j| \; P_{i_1,i_2,\ldots,(i_j-1),\ldots,i_K}(t) \\ & + \nu \sum_{j=\theta}^{N} |i_j| \; P_{i_1,i_2,\ldots,(i_j+1),\ldots,i_K}(t) \end{aligned} \tag{24}$$

At equilibrium, the left-hand side of (24) is zero. In theory, we can solve the system of equations (24) to obtain the steady-state probabilities for each market state. However, the practical difficulties of finding such a solution are considerable and we shall later turn to Monte Carlo methods for its approximation.

Some qualitative observations on the double-auction market structure are in order. First, we note that the dynamics (24) of such a structure are non-simple. In particular, the embedded transactions process is intimately linked to the market state process, and it would not be very surprising to find unusual dependencies in the former that derive from the complexities of the latter. (We specifically have in mind (a) the leptokurtosis observed in real-world price-change data, and (b) the findings of Niederhoffer and Osborne regarding serial dependencies in transaction-to-transaction price-change data.) Explorations of these dependencies will await future Monte Carlo studies.